Refresher Course on General Relativity
IUCAA (23 June - 5 July 2025)

Tutorial: Kerr Spacetime, Energy Extraction and Shadows

In this tutorial we will work in natural units, where $G = c = 1$.

1. **Basic Features of Kerr Metric:**
   The Kerr line element in Boyer-Lindquist coordinates is given by,

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right) dt^2 - \frac{4Mar\sin^2\theta}{\Sigma} dt\, d\phi + \frac{A\sin^2\theta}{\Sigma} d\phi^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma\, d\theta^2 \;, \quad (1)$$

   where $\Delta(r) \equiv r^2 - 2Mr + a^2$, $\Sigma \equiv r^2 + a^2\cos^2\theta$, $A \equiv \Sigma\Delta + 2Mr(r^2 + a^2)$.

   (a) Obtain the inner event horizon (r_-) and the outer event horizon (r_+) for $a < M$. Show that the Kretschmann scalar, given by

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = \frac{48M^2(r^2 - a^2\cos^2\theta)\left(\Sigma^2 - 16r^2a^2\cos^2\theta\right)}{\Sigma^6} \;, \quad (2)$$

   is finite at $r_\pm$, indicating coordinate singularity. Furthermore, find the true singularity.

   (b) Find the condition where $g_{tt} = 0$. What does a surface having $g_{tt} = 0$ signify, in general?

   (c) The region bounded by the two solutions above is called as Ergosphere. Show that in ergosphere, a massive particle cannot stay at a fixed Boyer-Lindquist coordinate, that is, with 4-velocity $u = (1, 0, 0, 0)$?

   (d) The Kerr metric in these coordinates has a conserved quantity, $l = g_{\phi\phi}u^\phi + g_{t\phi}u^t$, that we identify as the angular momentum of the particle. Using this, show that even for a zero angular momentum observer (ZAMO), the coordinate angular velocity is non-zero because of *frame-dragging*.

   (e) Find the limits within which the coordinate angular-velocity should remain.

   (f) Show that the event horizon is also a Killing horizon.

2. **Energy Extraction from Black Holes**

   (a) Using the first and the second law of black hole (BH) mechanics, which says *"In any process involving a black hole and other objects, the total energy, momentum, angular momentum and electric charge are conserved."*, and *"In any physical interaction, the surface area of a black hole can never decrease."*, respectively, prove that the energy can not be extracted from a Schwarzschild BH.

   (b) Perform a similar exercise for a Kerr BH with mass M and angular momentum $J = aM$, where $0 \leq a \leq M$, and show that the change in mass δM and angular momentum δJ must satisfy:

$$\delta M \geq \frac{a\delta J}{r_+^2 + a^2}, \quad (3)$$

   where r_+ is the outer event horizon. This implies that energy extraction from a Kerr BH is possible.

   (c) There exists a limit to how much energy can be extracted from a rotating BH, defined by the *irreducible mass* M_{irr}, which represents the minimum mass a Kerr BH can reach after all possible energy extraction. One can also think of the irreducible mass as the mass of the Schwarzschild BH with the same area for the event horizon as that of the Kerr BH. Show that, for a Kerr BH:

$$M_{irr}^2 = \frac{1}{2}(M^2 + \sqrt{M^4 - J^2}); \text{or, equivalently } M^2 = M_{irr}^2 + \left(\frac{J}{2M_{irr}}\right)^2 \quad (4)$$

(d) Using the above expression, show that the maximum extractable energy fraction from a Kerr BH is $(\sqrt{2}-1)/\sqrt{2}$, approximately 30% of its total energy. (For comparison, hydrogen fusion releases only $\sim 0.7\%$ of the total energy.)

(e) **Penrose Process**
Penrose process, named after Roger Penrose who proposed it in 1969, is one of the processes that can extract energy from a Kerr black hole. Suppose, a particle A reaches inside the ergosphere with 4-velocity u_A^α and splits into two fragments B and C. We let particle B to fall into the BH and particle C escape to infinity. Conservation of 4-momentum implies that $p_A^\mu = p_B^\mu + p_C^\mu$, so the energies (measured at infinity) must satisfy $E_A = E_B + E_C$. If we demand $E_B < 0$, we have $E_C > E_A$, and hence energy is extracted. Show that the the energy bounds of the fragmented particles must satisfy *Wald inequality*:

$$\gamma\left[\widetilde{E}_A - |v|\sqrt{\widetilde{E}_A^2 + g_{tt}}\right] \leq \widetilde{E} \leq \gamma\left[\widetilde{E}_A + |v|\sqrt{\widetilde{E}_A^2 + g_{tt}}\right], \tag{5}$$

where $\widetilde{E}$ represents the specific energy (energy per unit mass) and γ is the Lorentz factor. The lower bound will tell us the minimum energy of the fragmented particle B, which we will demand to be negative, and the upper bound will give us an idea of the maximum energy of the escaping particle C.

(f) Using the above inequality, show that if $g_{tt} > 0$, v can always be chosen large enough such that $E_B < 0$.

(g) Using the above inequality, argue why the Penrose process is not an astrophysically viable mechanism for energy extraction from rotating BHs. [Hint: Use the fact that $g_{tt} < 1$ outside the horizon.]

3. Consider a general static spherically symmetric metric

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{g(r)} + h(r)d\Omega^2. \tag{6}$$

(a) For geodesic motion show that $\theta = \pi/2$ is a stable plane, i.e., equatorial geodesics are planar. [Discussion: In fact, one can prove the planar nature of equatorial geodesics for a much larger class of stationary axis-symmetric spacetimes having a Z_2-symmetry, i.e. $g_{\mu\nu}(\theta) = g_{\mu\nu}(\pi - \theta)$.]

(b) Find out the light ring equation.

(c) Show that if the spacetime represents an asymptotically flat black hole there must be an odd number of light rings. [Discussion: Understand the radial stability of these light rings.]

(d) Show that if the spacetime represents an asymptotically flat horizonless compact object there will be an even number of light rings. Use the conditions: $f(r \to 0) > 0$ and $h(r \to 0) = 0$. [Discussion: Understand the radial stability of these light rings.]